CURVES IN BANACH SPACES WHICH ALLOW A C^2 PARAMETRIZATION

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Abstract. We give a complete characterization of those $f : [0,1] \to X$ (where X is a Banach space which admits an equivalent Fréchet smooth norm) which allow an equivalent C^2 parametrization. For $X = \mathbb{R}$, a characterization is well-known. However, even in the case $X = \mathbb{R}^2$, several quite new ideas are needed. Moreover, the very close case of parametrizations with a bounded second derivative is solved.

1. Introduction

Let X be a (real) Banach space, and a curve $f : [a,b] \to X$ be given. Several authors (e.g. Ward, Zahorski, Choquet, Tolstov) investigated conditions under which f allows an equivalent parametrization which is "smooth of the first order" (e.g., it is differentiable, boundedly differentiable, or continuously differentiable). Their results, which deal with the case $X = \mathbb{R}^n$, were generalized (using more or less difficult modifications of known methods) to the case of an arbitrary X in [DZ] and [D1], where information concerning the history of the "first order case" can be found.

The case of "higher order smooth" parametrizations for $X = \mathbb{R}$ was settled independently by Laczkovich and Preiss [LP] and Lebedev [L]. Both papers contain (formally slightly different, see Section 7 below) characterizations of those $f : [0,1] \to \mathbb{R}$ which allow an equivalent C^n ($n \in \mathbb{N}$) parametrization or a $C^{n,\alpha}$ ($0 < \alpha \le 1$) parametrization (i.e., a parametrization, whose n-th derivative is α-Hölder). The case of $f : [0,1] \to \mathbb{R}$ which allow an n-times differentiable parametrization was settled in [D2].

The problem of "higher order smooth" parametrizations in the vector case (even for $X = \mathbb{R}^2$) is essentially more difficult than in the case $X = \mathbb{R}$. In the present article we characterize those $f : [0,1] \to X$ which allow an equivalent C^2 parametrization if X admits an equivalent Fréchet smooth norm. We were not able to solve the vector-valued problem of C^n parametrization for $n \ge 3$ (even for $X = \mathbb{R}^2$). It seems to us that if there is a satisfactory solution of this problem, then it requires some new ideas.

An earlier (unpublished) preprint [DZ2] treated the case of C^2 parametrizations together with the case of parametrizations with bounded convexity. In the present article, the more interesting C^2 case is treated separately and with a simplified proof of a basic lemma (Lemma 3.6 below). Moreover, we also treat a very similar case of curves allowing a $D^{2,\infty}$ parametrization (i.e., parametrization with bounded second derivative), since we observed that the main lemmas (Lemma 3.6 and Lemma 3.17)

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can be proved for both cases simultaneously. The only new idea in the $D^{2,\infty}$ case is the definition of the notion of an $[f,\delta,K]$-partition (together with Lemma 3.10 and Lemma 3.12) which is more complicated than that of an (f,δ,K)-partition (which is sufficient for the C^2 case). Let us remark that the C^2 and $D^{2,\infty}$ cases are equivalent for $X=\mathbb{R}$ (see [LP, Remark 3.7] or Section 7 below), but are not equivalent for $X=\mathbb{R}^2$ (see Example 6.1 below).

An essential modification of our methods gives a full characterization of vector-valued curves allowing $C^{1,\alpha}$ $(0<\alpha\leq 1)$ parametrizations [DZ3]. Using another method (which follows [L] and [LP]), the case of of vector-valued curves allowing a twice differentiable parametrizations was settled in [D3].

The structure of the paper is the following. In Section 2 we recall some known notions and their well-known properties. In Section 3 we introduce the main special notions (the variation $W^{\delta}(f,G)$ and some types of generalized partitions of open sets) and prove a number of lemmas. The core of the article are Lemma 3.6 and Lemma 3.17. Section 4 contains main results on C^2 parametrizations. A consequence, which might be interesting also from the point of view of differential geometry, is presented in Proposition 4.9. In Section 5 the results concerning $D^{2,\infty}$ parametrizations are presented (the proofs which are almost literally same as those in Section 4 are omitted). Section 6 contains several examples, which show applicability of our results. In Section 7, we consider the case of real functions ($X=\mathbb{R}$); we show that in this case our conditions easily reduce to those found by Lebedev in [L].

2. Preliminaries

By λ we will denote the Lebesgue measure on $\mathbb{R}$ and all integrals are Lebesgue integrals. Throughout the whole article, X will always be a (real) Banach space. By $\mathcal{H}^1$ we will denote the 1-dimensional Hausdorff measure. The symbol $\sim$ is used for the strong equivalence (i.e., $f\sim g$ means $f/g\to 1$).

A mapping is *L-Lipschitz* provided it is Lipschitz with some constant L (not necessarily the minimal one). If $M\subset A\subset\mathbb{R}$ and $f:A\to X$ are given, then we define the *variation of f on M* as

$$V(f,M):=\sup\{\sum_{i=1}^{n}\|f(x_{i-1})-f(x_i)\|\},$$

where the supremum is taken over all $(x_i)_{i=0}^n\subset M$ such that $x_0<x_1<\cdots<x_n$. (We set $V(f,M):=0$, if M is empty or a singleton.) We say that $f:[a,b]\to X$ is *BV* (or *has bounded variation*), provided $V(f,[a,b])<\infty$.

For basic well-known properties of variation, see, e.g., [F] and [Chi]. In particular, we will need the additivity of variation (see [Chi, (P3) on p. 263]):

$$V(f,M)=V(f,M\cap(-\infty,t])+V(f,M\cap[t,\infty)),\quad \text{whenever}\quad t\in M. \tag{2.1}$$

If $f:[a,b]\to X$ is BV, then we define $v_f(x):=V(f,[a,x])$, $x\in[a,b]$. If f is also continuous, then v_f is continuous as well ([F], [Chi]). Moreover, clearly v_f is (strictly) increasing, if and only if f is not constant on any subinterval of $[a,b]$. We say that $f:[a,b]\to X$ *is parametrized by the arc-length*, if $V(f,[u,v])=v-u$ for every $a\leq u<v\leq b$. Obviously, each such f is 1-Lipschitz ([Chi, p. 267]).

Definition 2.1. (a) Let $f : [a,b] \to X$ be a continuous mapping. We say that $f^* : [c,d] \to X$ is a parametrization of f if there exists an increasing homeomorphism $h : [c,d] \to [a,b]$ such that $f^* = f \circ h$. If f^* is moreover parametrized by the arc-length, we say that f^* is an *arc-length parametrization of* f.

(b) If $f : [a,b] \to X$ is nonconstant, continuous and BV, then there exists (see [F, §2.5.16] or [Chi, Theorem 3.1]) a unique $F : [0,\ell] \to X$ (where $\ell := v_f(b)$) such that $f = F \circ v_f$. We will denote this associated mapping F by $\mathcal{A}_f$.

It is easy to see that $\mathcal{A}_f$ is always parametrized by the arc-length, and thus it is 1-Lipschitz (see [Chi]). We will use several times the following easy lemma.

Lemma 2.2. *Let $f : [a,b] \to X$ be continuous. Then the following hold.*

(i) *The function f has an arc-length parametrization if and only if f is BV and f is not constant on each $[c,d] \subset [a,b]$. In this case, $\mathcal{A}_f$ is an arc-length parametrization of f, $\mathcal{A}_f = f \circ (v_f)^{-1}$, and a general arc-length parametrization of f is of the form $F^s(x) = \mathcal{A}_f(x-s)$, $x \in [s, s+\ell]$, where $\ell := v_f(b)$ and $s \in \mathbb{R}$.*

(ii) *If f is BV on $[a,b]$, and is not constant on each subinterval of an interval $[\alpha,\beta] \subset [a,b]$, then $\mathcal{A}_f|_{[v_f(\alpha),v_f(\beta)]} = f \circ (v_f|_{[\alpha,\beta]})^{-1}$ is an arc-length parametrization of $f|_{[\alpha,\beta]}$.*

Let $f : [a,b] \to X$. The derivative f' is defined in the usual way; at the endpoints we take the corresponding unilateral derivatives. We say that $f : [a,b] \to X$ *is* C^n ($n \in \mathbb{N}$) provided the n-th derivative $f^{(n)}$ exists and is continuous on $[a,b]$. We will say that $f : [a,b] \to X$ *is* $D^{2,\infty}$ if f'' exists and is bounded on $[a,b]$. Clearly, if f is C^2, then f is $D^{2,\infty}$. Further, if f is $D^{2,\infty}$, then f' is clearly Lipschitz.

We will need also the following almost obvious lemma.

Lemma 2.3. *Let X be a Banach space and $f : [0,1] \to X$. Then the validity of the statement that f is C^2 (resp. $D^{2,\infty}$) does not depend on a choice of an equivalent norm on X.*

It is well known (see e.g. [VZ, p.2], or use [Ki, Theorem 7] together with [F, Theorem 2.10.13]) that if $f : [a,b] \to X$ is Lipschitz, and $f'(x)$ exists for almost all $x \in [a,b]$, then

$$V(f,[a,b]) = \int_a^b \|f'(x)\|\,dx. \tag{2.2}$$

For a proof of the following well-known version of Sard's Theorem, see e.g. [Ki, Theorem 7].

Lemma 2.4. *Let $f : [0,1] \to X$ be arbitrary. Let $C := \{x \in [0,1] : \ f'(x) = 0\}$. Then $\mathcal{H}^1(f(C)) = 0$.*

We will need also the following known lemmas.

Lemma 2.5. *If X is a Banach space, $f : [a,b] \to X$ is continuous, BV, not constant on any interval, and such that $F = f \circ (v_f)^{-1}$ is C^1, then $\|F'(x)\| = 1$ for each $x \in [0,\ell]$, where $\ell = v_f(b)$.*

Proof. Since F is an arc-length parametrization of f, and thus 1-Lipschitz, we obtain $\|F'(s)\| \leq 1$, $s \in [0,\ell]$, and $\ell = V(F,[0,\ell]) = \int_0^{\ell} \|F'(s)\|\, ds$ by (2.2). Since F is C^1, we obtain $\|F'(s)\| = 1$ for each $s \in [0,\ell]$. $\square$

Lemma 2.6. *Let $f : [a,b] \to X$ be continuous. Let $G \subset (a,b)$ be an open set, $H := [a,b] \setminus G$ and (a_t,b_t), $t \in T$, be all (pairwise different) components of G. Then:*

(i) *If $\mathcal{H}^1(f(H)) = 0$, then $V(f,[a,b]) = \sum_{t\in T} V(f,[a_t,b_t])$.*
(ii) *If $V(f,[a,b]) = \sum_{t\in T} V(f,[a_t,b_t]) < \infty$, then $\mathcal{H}^1(f(H)) = 0$.*
(iii) *If $\mathcal{H}^1(f(H)) = 0$ and f is L-Lipschitz on each $[a_t,b_t]$, then f is L-Lipschitz on $[a,b]$.*
(iv) *If f is BV and $\mathcal{H}^1(f(H)) = 0$, then $\lambda(v_f(H)) = 0$.*

Proof. Part (i) is an easy consequence of the vector form ([F, Theorem 2.10.13]) of Banach indicatrix theorem; see [DZ, Lemma 2.7]. For part (ii), let $\varepsilon > 0$ and choose a finite $S \subset T$ such that $\sum_{t\in S} V(f,[a_t,b_t]) > V(f,[a,b]) - \varepsilon$. Then by [F, Corollary 2.10.12] we see that

$$\mathcal{H}^1(f(H)) \leq \sum_{i=1}^{N} \mathcal{H}^1(f([c_i,d_i])) \leq \sum_{i=1}^{N} V(f,[c_i,d_i]) \leq \varepsilon,$$

where $[c_i,d_i]$, $i = 1,\dots,N$, are components of $[a,b] \setminus \bigcup_{t\in S}(a_t,b_t)$. Therefore $\mathcal{H}^1(f(H)) = 0$.

For part (iii), consider arbitrary $c,d \in H$, $c < d$. Applying (i) to f on $[c,d]$, we obtain

$$\begin{aligned} |f(d) - f(c)| \leq V(f,[c,d]) &= \sum \{V(f,[a_t,b_t]) : t \in T, [a_t,b_t] \subset [c,d]\} \\ &\leq \sum \{L|b_t - a_t| : t \in T, [a_t,b_t] \subset [c,d]\} \leq L(d-c), \end{aligned}$$

where we used the fact that the variation of an L-Lipschitz function on an interval is also L-Lipschitz. Since f is L-Lipschitz on H and on each $[a_t,b_t]$, it is clearly L-Lipschitz on $[a,b]$. Part (iv) easily follows from (i). $\square$

For the following well-known fact see [B, Theorem 2.3, p. 35].

Lemma 2.7. *Let $C \subset [0,1]$ be a closed nowhere dense set. Then there exists a real function φ on $[0,1]$ which has a bounded derivative on $[0,1]$, and C is the set of points of discontinuity of φ'.*

3. Basic special notions and lemmas

The following easy inequality is well known (see e.g. [MS, Lemma 5.1]):

$$\text{if } u,v \in X \setminus \{0\}, \text{ then } \left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leq \frac{2}{\|u\|}\,\|u - v\|. \tag{3.1}$$

Lemma 3.1. *Let I be a closed interval and $f : I \to X$ be such that $\|f'(x)\| \geq i > 0$ and $\|f''(x)\| \leq M$ for each $x \in I$. Let F be an arc-length parametrization of f, and suppose that $F''(z)$ exists. Then $\|F''(z)\| \leq \frac{2\,M}{i^2}$.*

Proof. By Lemma 2.2(i), we can suppose that $F = f \circ v_f^{-1}$. Let $x, y \in I$. Since f' is M-Lipschitz, by (3.1) we have

$$\left\| \frac{f'(x)}{\|f'(x)\|} - \frac{f'(y)}{\|f'(y)\|} \right\| \le \frac{2}{\|f'(x)\|} \|f'(x) - f'(y)\| \le \frac{2\,M}{i} |x-y|.$$

Consequently, $g := \frac{f'}{\|f'\|}$ is $\frac{2M}{i}$-Lipschitz. Since (2.2) implies $(v_f)' = \|f'\|$, we have $(v_f^{-1})' = \frac{1}{\|f' \circ v_f^{-1}\|}$. Thus, v_f^{-1} is $\frac{1}{i}$-Lipschitz; so $F' = (f \circ v_f^{-1})' = g \circ v_f^{-1}$ is $\frac{2M}{i^2}$-Lipschitz on $v_f(I)$, and the conclusion follows. □

We say that a Banach space X *has a Fréchet smooth norm*, provided the norm $\|\cdot\|$ of X is Fréchet differentiable at all $x \in X \setminus \{0\}$ (or equivalently $\|\cdot\|$ is C^1 on $X \setminus \{0\}$).

Lemma 3.2. *Let X be a Banach space which admits an equivalent Fréchet smooth norm. Let I be a closed interval and $f : I \to X$ be such that $f'(x) \neq 0$ for each $x \in I$. If f is $D^{2,\infty}$ (resp. C^2), then $F := f \circ v_f^{-1}$ is $D^{2,\infty}$ (resp. C^2).*

Proof. We can assume that X has a Fréchet smooth norm by Lemma 2.3. Let f be $D^{2,\infty}$. Since clearly $i := \min\{\|f'(x)\| : \ x \in I\} > 0$, by Lemma 3.1 it is sufficient to observe that $F''(y)$ exists for each $y \in v_f(I)$. In the proof of Lemma 3.1 we obtained the equality $(v_f^{-1})' = \frac{1}{\|f' \circ v_f^{-1}\|}$, which implies that $(v_f^{-1})'$ is differentiable. So, v_f^{-1} is twice differentiable, which implies that $F := f \circ v_f^{-1}$ is twice differentiable.

If f is C^2, then the above equality implies that $(v_f^{-1})'$ is C^1. So, v_f^{-1} is C^2, which implies that $F := f \circ v_f^{-1}$ is C^2. □

Now we define a special type of an "1/2-variation", which is crutial in our solution of C^2 and $D^{2,\infty}$ parametrization problems.

Definition 3.3. Let $f : [0,1] \to X$ be continuous and BV. Let $\emptyset \neq G \subset (0,1)$ be an open set and $0 < \delta < \infty$. Suppose that f is not constant on each interval contained in G and $F := \mathcal{A}_f$ (see Definition 2.1) is twice differentiable on $v_f(G)$. Then we define

$$W^\delta(f, G) = \sup \Big\{ \sum_{k=1}^{n} \ \sqrt{V(f, I_k)} \ \Big\},$$

where the supremum is taken over all non-overlapping systems $I_1, \dots, I_n$ of compact intervals with $\operatorname{int}(I_k) \subset G$ such that $S_{I_k} \cdot V(f, I_k) \ge \delta$ whenever $I_k \subset G$ (where $S_{I_k} = \sup_{x \in v_f(\operatorname{int}(I_k))} \|F''(x)\| \le \infty$).

Remark 3.4. Let f, G and $\delta > 0$ be as in Definition 3.3, and let $\omega : [0,1] \to [0,1]$ be an increasing homeomorphism. Then

$$W^\delta(f, G) = W^\delta(f \circ \omega, \omega^{-1}(G)).$$

This equality easily follows from the definition, if we observe that $\mathcal{A}_f = \mathcal{A}_{f \circ \omega}$ and $v_{f \circ \omega} = v_f \circ \omega$.

Remark 3.5. Let f, G and $\delta > 0$ be as in Definition 3.3, and let $\mathcal{I}$ be the family of all components of G. Then, using only Definition 3.3, we clearly obtain

$$\sum_{I \in \mathcal{I}} \sqrt{V(f, I)} \le W^\delta(f, G).$$

.

Our first basic lemma follows.

Lemma 3.6. *Suppose that $f:[0,1]\to X$ is continuous and BV, and f'' exists and is bounded on $(0,1)$. Let $\emptyset\neq G\subset(0,1)$ be an open set such that $f'(e)=0$, whenever $e\in(0,1)$ is an endpoint of any component of G. Let f be nonconstant on each interval contained in G and let $F:=\mathcal{A}_f$ (see Definition 2.1) be twice differentiable on $v_f(G)$. Then, $W^\delta(f,G)<\infty$ for each $\delta>0$.*

Proof. Denote $M:=\sup_{x\in(0,1)}\|f''(x)\|$. Choose $\delta>0$ and consider a system $I_1,\dots,I_n$ of non-overlapping compact intervals with $\operatorname{int}(I_k)\subset G$ such that $S_{I_k}\cdot V(f,I_k)\geq\delta>0$ whenever $I_k\subset G$ (where $S_{I_k}=\sup_{x\in v_f(\operatorname{int}(I_k))}\|F''(x)\|$).

Further, consider an arbitrary $I_k=:I$ such that $I\subset(0,1)$. Denote $i:=\min\{\|f'(x)\|:x\in I\}$ and choose $\tilde{x}\in I$ such that $\|f'(\tilde{x})\|=i$. The Mean Value Theorem implies that, for each $x\in I$,

$$\|f'(x)\|-i\leq\|f'(x)-f'(\tilde{x})\|\leq M\lambda(I).$$

So, $\|f'(x)\|\leq i+M\lambda(I)$, and thus (2.2) implies

$$V(f,I)=\int_I\|f'(x)\|\,dx\leq i\lambda(I)+M(\lambda(I))^2. \tag{3.2}$$

Thus, if $i=0$, we obtain

$$\sqrt{V(f,I)}\leq\sqrt{M}\,\lambda(I). \tag{3.3}$$

If $i>0$, then the assumptions of the lemma imply that $I\subset G$, and therefore $\sup_{x\in v_f(\operatorname{int}(I))}\|F''(x)\|\,V(f,I)\geq\delta$. By Lemma 3.1 and Lemma 2.2(ii), we obtain

$$\frac{2M}{i^2}\,V(f,I)\geq\delta,\quad\text{and so}\quad i\leq\sqrt{\frac{2M}{\delta}V(f,I)}. \tag{3.4}$$

Using (3.2), (3.4) and the A-G inequality, we obtain subsequently

$$V(f,I)\leq\sqrt{\frac{2M}{\delta}V(f,I)}\,\lambda(I)+M(\lambda(I))^2=\sqrt{V(f,I)\cdot\frac{2M}{\delta}(\lambda(I))^2}+M(\lambda(I))^2,$$

$$V(f,I)\leq\frac{V(f,I)}{2}+\frac{M}{\delta}(\lambda(I))^2+M(\lambda(I))^2,$$

$$\sqrt{V(f,I)}\leq\sqrt{2\left(\frac{M}{\delta}+M\right)}\,\lambda(I). \tag{3.5}$$

Since there are at most two intervals $I\in\{I_1,\dots,I_n\}$ which are not contained in $(0,1)$, and $\lambda(I_1)+\cdots+\lambda(I_n)\leq 1$, we obtain by (3.3) and (3.5)

$$\sum_{k=1}^n\sqrt{V(f,I_k)}\leq\sqrt{M}+\sqrt{2\left(\frac{M}{\delta}+M\right)}+2\sqrt{V(f,[0,1])}.$$

Consequently, $W^\delta(f,G)<\infty$. □

We will work many times with the following natural notion of a generalized partition of an open set.

Definition 3.7. (i) We say that $\mathcal{I}\subset\mathbb{Z}$ is an $\mathbb{Z}$-interval, if $\mathcal{I}=(l,m)\cap\mathbb{Z}$, where $l,m\in\mathbb{Z}^*=\mathbb{Z}\cup\{-\infty,\infty\}$.

(ii) We will say that a family $\mathcal{P}$ of compact intervals is a *generalized partition of a bounded interval* (a,b), if there exists a system $(x_i)_{i\in\mathcal{I}}$ such that $\mathcal{I}$ is an $\mathbb{Z}$-interval, the function $i \mapsto x_i,\ i \in \mathcal{I}$, is strictly increasing, $\inf_{i\in\mathcal{I}} x_i = a$, $\sup_{i\in\mathcal{I}} x_i = b$ and $\mathcal{P} = \{[x_k, x_{k+1}] :\ k, k+1 \in \mathcal{I}\}$.

(iii) We will say that a family $\mathcal{P}$ of compact intervals is a *generalized partition of a bounded open set* $\emptyset \neq G \subset \mathbb{R}$, if $\bigcup\{\mathrm{int}(I) : I \in \mathcal{P}\} \subset G$, and for each component (a,b) of G, the family $\{I \in \mathcal{P} :\ \mathrm{int}(I) \subset (a,b)\}$ is a generalized partition of (a,b).

In fact, we will work only with special generalized partitions. In the $D^{2,\infty}$ problem, we need the notion of an $[f,\delta,K]$-partition. In the $C^{2,\infty}$ problem, it is sufficient to work only with the more special (and simpler) notion of an (f,δ,K)-partition.

Definition 3.8. Suppose that $f : [0,1] \to X$ is continuous and BV. Further suppose that f is not constant on any interval which is a subset of an open set $\emptyset \neq G \subset (0,1)$. Set $F := \mathcal{A}_f$ (see Definition 2.1) and suppose that F'' exists on $v_f(G)$. For each compact interval L with $\mathrm{int}(L) \subset G$, set $S_L := \sup_{x\in v_f(\mathrm{int}(L))} \|F''(x)\|$. Let $\mathcal{P}$ be a generalized partition of G, $0 \le \delta \le K \le \infty$, and $\delta \in \mathbb{R}$. Then we say that $\mathcal{P}$ is an $[f,\delta,K]$*-partition of* G, if:

(a) $S_I < \infty$ and $S_I \cdot V(f,I) \le K$ for each $I \in \mathcal{P}$.

(b) For each $I \in \mathcal{P}$ with $I \subset G$, there exists $J \in \mathcal{P}$ such that $I \cap J \neq \emptyset$ and $S_{I\cup J} V(f, I\cup J) \ge \delta$.

We say that $\mathcal{P}$ is an (f,δ,K)*-partition of* G, if (a) holds, and

(b*) $S_I \cdot V(f,I) \ge \delta$ for each $I \in \mathcal{P}$ with $I \subset G$.

Remark 3.9. Clearly each (f,δ,K)-partition of G is an $[f,\delta,K]$-partition of G. The notions of an $[f,0,K]$-partition and of an $(f,0,K)$-partition coincide.

Lemma 3.10. *Let $\emptyset \neq G \subset (0,1)$ be open, and $f : [0,1] \to X$ be a continuous BV function. Let f be not constant on any subinterval of G, and let $F := \mathcal{A}_f$ (see Definition 2.1) have locally bounded second derivative (resp. be C^2) on $v_f(G)$. Then, for each $0 < \delta < \infty$, there exists a generalized partition $\mathcal{P}$ of G, which is an $[f,\delta,\delta]$-partition (resp. (f,δ,δ)-partition) of G.*

Proof. Without any loss of generality, we can assume that $G = (a,b) \subset (0,1)$. Let $x_0 := \frac{a+b}{2}$. We will construct points x_i $(i \in \mathbb{Z})$ with

$$a \le \cdots \le x_{-2} \le x_{-1} < x_0 < x_1 \le x_2 \le x_3 \le \cdots \le b.$$

Suppose that x_{n-1} $(n \in \mathbb{N})$ is defined. If $x_{n-1} = b$, then put $x_n = b$. If $x_{n-1} < b$, then set $x_n := \inf M_n$, where

$$M_n := \{t \in (x_{n-1}, b] :\ V(f,[x_{n-1},t]) \cdot \sup_{x\in v_f(x_{n-1},t)} \|F''(x)\| \ge \delta\}\ \cup \{b\}.$$

Since F'' is locally bounded on (a,b), we easily see that $x_n > x_{n-1}$. Further, it is easy to see that

$$V(f,[x_{n-1},x_n]) \cdot \sup_{x\in v_f(x_{n-1},x_n)} \|F''(x)\| \le \delta. \tag{3.6}$$

(Otherwise , using the definition of the supremum, we easily see that there exists a $t \in M_n \cap (x_{n-1}, x_n)$.)

We define the points x_n $(n < 0)$ quite symmetrically and set $\mathcal{P} := \{[x_i, x_{i+1}] : i \in \mathbb{Z}, x_i < x_{i+1}\}$. Since F'' is locally bounded on (a, b), it is easy to show that $\inf_{n \in \mathcal{J}} x_n = a$ and $\sup_{n \in \mathcal{J}} x_n = b$. So, $\mathcal{P}$ is a generalized partition of (a, b). To prove that $\mathcal{P}$ is an $[f, \delta, \delta]$-partition of (a, b), first observe that (3.6) holds whenever $[x_{n-1}, x_n] \in \mathcal{P}$ (also if $n - 1 < 0$), and so the condition (a) of Definition 3.8 holds. To prove also (b), consider an interval $I \in \mathcal{P}$ such that $I \subset (a, b)$, i.e. $x_{n-1} \neq a$ and $x_n \neq b$. If $n - 1 \geq 0$, then clearly $x_{n+1} \in M_n$. So, putting $J := [x_n, x_{n+1}]$, we obtain $S_{I \cup J} V(f, I \cup J) \geq \delta$. Similarly, if $n - 1 < 0$, we can set $J := [x_{n-2}, x_{n-1}]$.

In the case that F is C^2 on $v_f(G)$, then $\mathcal{P}$ is even an (f, δ, δ)-partition of (a, b). Indeed, let $I = [x_{n-1}, x_n]$, $n - 1 \geq 0$, and $x_n \neq b$. Using continuity of F'' in $v_f(x_n)$, we easily obtain $S_I \cdot V(f, I) = \delta$. Using a symmetrical argument, we obtain that the condition (b*) of Definition 3.8 holds.

□

Lemma 3.11. *Let a_i $(i \in I)$, b_j, c_j $(j \in J)$ be non-negative numbers, I countable, and J finite. Then*

$$\sqrt{\sum_{i \in I} a_i} \leq \sum_{i \in I} \sqrt{a_i} \quad \text{and} \quad \sum_{j \in J} \sqrt{b_j c_j} \leq \sqrt{\sum_{j \in J} b_j \cdot \sum_{j \in J} c_j}. \tag{3.7}$$

Proof. The first inequality is clear. The second is an immediate consequence of the Cauchy-Schwartz inequality. □

Lemma 3.12. *Let $f : [0, 1] \to X$ be continuous and BV. Let $\emptyset \neq G \subset (0, 1)$ be an open set and $0 < \delta < \infty$. Let f be nonconstant on each interval contained in G, let $F := \mathcal{A}_f$ (see Definition 2.1) have locally bounded second derivative on $v_f(G)$, and let $W^\delta(f, G) < \infty$. Then the following hold.*

(i) *If $\mathcal{P}$ is an $[f, \delta, \infty]$-partition of G, then $\sum_{I \in \mathcal{P}} \sqrt{V(f, I)} < \infty$.*
(ii) $\int_{v_f(G)} \sqrt{\|F''\|} < \infty$.

Proof. Let $\mathcal{P}$ be an $[f, \delta, \infty]$-partition of G. For each compact interval L with $\mathrm{int}(L) \subset G$, set $S_L := \sup_{x \in v_f(\mathrm{int}(L))} \|F''(x)\|$. Set $\mathcal{P}' = \{I \in \mathcal{P} : I \subset G\}$ and $\mathcal{P}'' = \mathcal{P} \setminus \mathcal{P}'$. By Definition 3.3, we have

$$\sum_{I \in \mathcal{P}''} \sqrt{V(f, I)} \leq W^\delta(f, G) < \infty \tag{3.8}$$

By Definition 3.8, for each $I \in \mathcal{P}'$ we can choose an interval $J =: n(I)$ such that

$$V(f, I \cup n(I)) \cdot S_{I \cup n(I)} \geq \delta.$$

It is easy to see that $\{I \cup n(I) : I \in \mathcal{P}'\}$ can be written as $\bigcup_{i=1}^3 \mathcal{P}_i$, where $\mathcal{P}_i$ is a family of non-overlapping compact intervals for each $i \in \{1, 2, 3\}$. Thus,

$$\sum_{I \in \mathcal{P}'} \sqrt{V(f, I)} \leq \sum_{i=1}^{3} \sum_{I \in \mathcal{P}_i} \sqrt{V(f, I \cup n(I))} \leq 3W^\delta(f, G) < \infty.$$

So, (3.8) implies $\sum_{I \in \mathcal{P}} \sqrt{V(f, I)} < \infty$. Thus, we have proved (i).

To prove (ii), choose by Lemma 3.10 an $[f,\delta,\delta]$-partition $\mathcal{P}$ of G. Observe that $\|F''\|$ is Lebesgue measurable. Thus

$$\begin{aligned}\int_{v_f(G)} \sqrt{\|F''\|} &\leq \sum_{I\in\mathcal{P}} \lambda(v_f(I)) \cdot \sup_{x\in v_f(\mathrm{int}(I))} \sqrt{\|F''(x)\|} \\ &= \sum_{I\in\mathcal{P}} \sqrt{V(f,I)} \cdot \sup_{x\in v_f(\mathrm{int}(I))} \sqrt{\|F''\|\cdot V(f,I)} \leq \sqrt{\delta}\sum_{I\in\mathcal{P}}\sqrt{V(f,I)} < \infty,\end{aligned}$$

where the last inequality follows from (i). □

Lemma 3.13. *Let $f:[0,1]\to X$ be continuous and BV. Let $\emptyset \neq G \subset (0,1)$ be an open set such that f is nonconstant on any interval contained in G and let $F := \mathcal{A}_f$ (see Definition 2.1) have locally bounded second derivative on $v_f(G)$. Suppose that $\mathcal{S}$ is a family of pairwise non-overlapping compact intervals such that* $\mathrm{int}(J)\subset G$ *for each $J\in\mathcal{S}$ and*

$$\sum_{J\in\mathcal{S}} V(f,J) = V(f,[0,1]),\ \sum_{J\in\mathcal{S}}\sqrt{V(f,J)} < \infty,\ \sum_{J\in\mathcal{S}} V(f,J)\sqrt{S_J} < \infty, \tag{3.9}$$

where $S_J = \sup_{y\in v_f(\mathrm{int}(J))}\|F''(y)\|$. Then $W^\delta(f,G)<\infty$ for each $\delta>0$.

Proof. Let $\delta>0$ and consider a finite system $\mathcal{K}$ of non-overlapping compact intervals such that $\mathrm{int}(I)\subset G$ for each $I\in\mathcal{K}$, and $S_I\cdot V(f,I)\geq\delta$ whenever $I\in\mathcal{K}$ and $I\subset G$. For each $J\in\mathcal{S}$, let $\mathcal{K}_J := \{I\in\mathcal{K} : I\subset \mathrm{int}(J)\}$. Set $\mathcal{K}_1 := \bigcup\{\mathcal{K}_J : J\in\mathcal{S}\}$ and $\mathcal{K}_2 := \mathcal{K}\setminus\mathcal{K}_1$. For each $J\in\mathcal{S}$, we obtain

$$\sqrt{\delta}\sum\{\sqrt{V(f,I)} : I\in\mathcal{K}_J\} \leq \sum\{\sqrt{S_I}\cdot V(f,I) : I\in\mathcal{K}_J\} \leq \sqrt{S_J}\cdot V(f,J).$$

Therefore

$$\sum\{\sqrt{V(f,I)} : I\in\mathcal{K}_1\} \leq (1/\sqrt{\delta})\sum\{\sqrt{S_J}\cdot V(f,J) : J\in\mathcal{S}\}. \tag{3.10}$$

For each $I\in\mathcal{K}_2$, denote by $\mathcal{S}_I$ the set of all $J\in\mathcal{S}$, such that $J\cap\mathrm{int}(I)\neq\emptyset$. If $I=[a,b]\in\mathcal{K}_2$, put $a^* := \min J_a$, if there exists $J_a\in\mathcal{S}$ with $a\in\mathrm{int}(J_a)$ and $a^* := a$, if such J_a does not exist. Similarly, put $b^* := \max J_b$, if there exists $J_b\in\mathcal{S}$ with $b\in\mathrm{int}(J_b)$ and $b^* := b$, if such J_b does not exist. The equality of (3.9) easily implies that $V(f,[a^*,b^*]) = \sum\{V(f,J) : J\in\mathcal{S}_I\}$. (It can be shown either directly, or using first Lemma 2.6(ii) and then Lemma 2.6(i).) Thus the first inequality of (3.7) implies $\sqrt{V(f,[a^*,b^*])}\leq\sum\{\sqrt{V(f,J)} : J\in\mathcal{S}_I\}$. Observing that, for each $J\in\mathcal{S}$, the set $\{I\in\mathcal{K}_2 : J\in\mathcal{S}_I\}$ contains at most two intervals, we obtain

$$\sum\{\sqrt{V(f,I)} : I\in\mathcal{K}_2\} \leq 2\sum\{\sqrt{V(f,J)} : J\in\mathcal{S}\}. \tag{3.11}$$

Now (3.9), (3.10) and (3.11) imply $W^\delta(f,G)<\infty$. □

Lemma 3.14. *Let $(I_\alpha)_{\alpha\in A}$ be a system of pairwise non-overlapping compact subintervals of an interval $[0,d]$. Let $\sum_{\alpha\in A}\mu_\alpha<\infty$, where $\mu_\alpha>0$, $\alpha\in A$. Then there exists an interval $[0,d']$ and an increasing homeomorphism $\Psi:[0,d']\to[0,d]$ such that $\lambda(\Psi^{-1}(I_\alpha))=\mu_\alpha$ and Ψ^{-1} is absolutely continuous.*

Proof. We can define $\Psi := \omega^{-1}$, where $\omega(x) = \int_0^x\varphi$ $(x\in[0,d'])$, and $\varphi(t) = \mu_\alpha/\lambda(I_\alpha)$ for $t\in\mathrm{int}(I_\alpha)$ and $\varphi(t)=1$ for $t\in[0,d']\setminus\bigcup\{\mathrm{int}(I_\alpha) : \alpha\in A\}$. □

Lemma 3.15. *Let $0 < d < 1$, $\xi > 0$ and $c_l, c_r \geq 0$ with $0 < \max(c_l, c_r) \leq 10^{-9} \cdot \xi d$ be given. Then, for each $I = [u, v]$ with $\lambda(I) = d$, there exists a C^2 function ω on I such that $\omega(u) = \omega(v) = 0$, $\omega'(x) = c_l$ for $x \in [u, u + d/3]$, $\omega'(x) = c_r$ for $x \in [u + 2d/3, v]$, and $\max(|\omega'(x)|, |\omega''(x)|) \leq \xi$ for $x \in I$.*

Proof. We can suppose that $I = [0, d]$. First assume that $c_l \leq c_r$. Denote $p_1 := 2(9c_r - c_l)\xi^{-1}$, $p_2 := 20 \cdot c_r \cdot \xi^{-1}$; clearly

$$0 < p_1 \leq p_2 < 10^{-7} d. \tag{3.12}$$

For every $0 \leq s \leq d/3 - p_1 - p_2$, let $x_0 := 0$, $x_1 := d/3$, $x_2 := x_1 + p_1/2$, $x_3 := x_2 + p_1/2$, $x_4 := x_3 + s$, $x_5 := x_4 + p_2/2$, $x_6 := x_5 + p_2/2$, $x_7 := d$. Clearly $x_i \leq x_{i+1}$ $(0 \leq i \leq 6)$. Now consider the function $\mu = \mu_s$ which is linear on each $[x_i, x_{i+1}]$ (if $x_i < x_{i+1}$) and

$$\mu(x_0) = \mu(x_1) = \mu(x_3) = \mu(x_4) = \mu(x_6) = \mu(x_7) = 0, \ \mu(x_2) = -\xi, \ \mu(x_5) = \xi.$$

Clearly $|\mu(x)| \leq \xi$, $x \in [0, d]$. Let $\nu(x) = \nu_s(x) = c_l + \int_0^x \mu$, $x \in I$. It is easy to see that $\nu(x) = c_l$ for $x \in [x_0, x_1]$, $\nu(x) = -9c_r$ for $x \in [x_3, x_4]$, $\nu(x) = c_r$ for $x \in [x_6, x_7]$, and $|\nu(x)| \leq 9c_r \leq \xi$ for $x \in I$. These properties of ν and (3.12) easily imply that $g(s) := \int_0^d \nu_s > 0$ for $s = 0$ and $g(s) < 0$ for $s = d/3 - p_1 - p_2$. Since $g(s)$ is clearly continuous, we can choose $s_0 \in (0, d/3 - p_1 - p_2)$ with $g(s_0) = 0$. Now it is easy to see that $\omega(x) := \int_0^x \nu_{s_0}$, $x \in I$, has all desired properties.

If $c_l > c_r$, we apply the just-proven assertion to $I^* := [-d, 0]$, $c_l^* := c_r$, $c_r^* := c_l$ and obtain a function ω^* on I^*. Now it is sufficient to put $\omega(x) := -\omega^*(-x)$, $x \in [0, d]$. $\square$

Lemma 3.16. *Let $V > 0$, $\eta > 0$ and $c_l, c_r \geq 0$ such that $d := \sqrt{V/\eta} < 1$ and $\max(c_l, c_r) \leq 10^{-10} \cdot d^2 \eta$ be given. Then, for every interval J of the length V and every interval $I = [u, v]$ of the length d, there exists an increasing C^2 homeomorphism $\varphi : I \to J$ such that $\varphi'(u) = c_l$, $\varphi'(v) = c_r$, $\varphi''(u) = \varphi''(v) = 0$, $0 < \varphi'(x) \leq 19 \cdot \sqrt{\eta V}$ for $x \in \operatorname{int}(I)$, and $|\varphi''(x)| \leq 19 \cdot \eta$ for $x \in I$.*

Proof. We can suppose that $J = [0, V]$ and $I = [0, d]$. Consider the function π on I which is linear on each $[id/6, (i+1)d/6]$ $(i = 0, \dots, 5)$, $\pi(d/6) = \eta$, $\pi(5d/6) = -\eta$ and $\pi(id/6) = 0$ for $i = 0, 2, 3, 4, 6$. Define $\rho(x) := \int_0^x \pi$, $x \in I$, and $\tau(x) := \int_0^x \rho$, $x \in I$. Clearly $\rho(0) = \rho(d) = 0$, $0 < \rho(x) \leq \eta d$ for $x \in (0, d)$ and $\rho(x) = d\eta/6$ for $x \in [d/3, 2d/3]$. Therefore $d^2\eta/18 \leq \tau(d) \leq \eta d^2$. Thus, setting $\Psi(x) = (\eta d^2/\tau(d))\tau(x)$ $(x \in I)$, we have $\Psi(d) = \eta d^2 = V$. Consequently, $\Psi : I \to J$ is an increasing homeomorphism with $\Psi' > 0$ on $\operatorname{int}(I)$. Further

$$|\Psi''(x)| \leq 18 \cdot |\tau''(x)| \leq 18 \cdot \eta, \quad \Psi'(x) \leq 18 \cdot \tau'(x) \leq 18 \cdot \eta d = 18\sqrt{\eta V} \quad (x \in I), \tag{3.13}$$

$$\Psi'(0) = \Psi'(d) = 0, \quad \text{and} \quad \Psi'(x) \geq \tau'(x) = \rho(x) = \frac{d\eta}{6}, \quad x \in \left[\frac{d}{3}, \frac{2d}{3}\right]. \tag{3.14}$$

Therefore, in the case $c_l = c_r = 0$, the function $\varphi := \Psi$ has the desired properties.

If $\max(c_l, c_r) > 0$, set $\xi := d\eta/7$. Then $\max(c_l, c_r) \leq 10^{-10} \cdot d^2\eta \leq 10^{-9} \cdot \xi d$. Thus we can use Lemma 3.15 and obtain a corresponding function ω on $[0, d]$. Define $\varphi := \Psi + \omega$. Since $|\omega'(x)| \leq d\eta/7$ if $x \in [d/3, 2d/3]$ and $\omega'(x) \geq 0$ otherwise, using (3.14) we obtain that $\varphi' > 0$ on $\operatorname{int}(I)$. Since $\omega(0) = \omega(d) = 0$, we obtain that φ is a homeomorphism of I onto J. Since $\omega'(0) = c_l$, $\omega'(d) = c_r$, $\max(|\omega'(x)|, |\omega''(x)|) \leq \xi$

for $x \in I$, and $\xi = \sqrt{V\eta}/7 = d\eta/7 < \eta$, we obtain by (3.13) and (3.14) that φ has all desired properties. $\square$

Lemma 3.17. *Suppose that $f : [0,1] \to X$ is BV continuous, $\emptyset \neq G \subset (0,1)$ is an open set, f is not constant on any subinterval of G and $F := \mathcal{A}_f$ (see Definition 2.1) has locally bounded second derivative on $v_f(G)$. Let $\mathcal{H}^1(f(H)) = 0$, where $H := [0,1] \setminus G$. Suppose that $0 < K < \infty$ and $\mathcal{P}$ is an $(f, 0, K)$-partition of G such that $\sum_{I \in \mathcal{P}} \sqrt{V(f,I)} < \infty$. Then there exists a homeomorphism $h : [0,1] \to [0,1]$ such that $f \circ h$ is $D^{2,\infty}$ and $(f \circ h)'(x) \neq 0$ for each $x \in h^{-1}(G)$. Moreover, if even*

$$F \text{ is } C^2 \text{ on } v_f(G), \tag{3.15}$$

then $f \circ h$ is C^2.

Further, in both cases, if f is nonconstant on any interval, then $\lambda(h^{-1}(H)) = 0$.

Proof. Let U be the maximal open set on which f is locally constant. Set $v^*(x) := v_f(x) + \lambda([0,x] \cap U)$, $x \in [0,1]$. It is clear that v^* is continuous and increasing. Put $d_1 := v^*(1)$ and $\xi := (v^*)^{-1}$; clearly $\xi : [0, d_1] \to [0,1]$ is an increasing homeomorphism. Denote $f_1 := f \circ \xi$, $G_1 := \xi^{-1}(G)$ and $\mathcal{P}_1 := \{\xi^{-1}(I) : I \in \mathcal{P}\}$. Clearly

$$\sum_{J \in \mathcal{P}_1} \sqrt{\lambda(J)} = \sum_{I \in \mathcal{P}} \sqrt{V(f,I)} < \infty. \tag{3.16}$$

If (γ, δ) is a component of G_1, then $f_1|_{[\gamma,\delta]}$ is clearly an arc-length parametrization of $f|_{[\xi(\gamma),\xi(\delta)]}$. So, using our assumptions and Lemma 2.2, we obtain that f_1 has locally bounded second derivative on G_1, and f_1 is C^2 on G_1, if (3.15) holds. Further, using Lemma 2.2, Lemma 2.5, and the properties of $\mathcal{P}$, we obtain that

$$\|f_1'(x)\| = 1 \quad \text{for} \quad x \in G_1, \quad \text{and} \tag{3.17}$$

$$\sup_{s \in \operatorname{int}(J)} \|f_1''(s)\| \cdot \lambda(J) \leq K \qquad \text{for each} \quad J \in \mathcal{P}_1. \tag{3.18}$$

Since (3.16) holds, it is easy to see that to each $J \in \mathcal{P}_1$ we can assign a number $\eta_J > 0$ such that

$$\sum_{J \in \mathcal{P}_1} \sqrt{\lambda(J)/\eta_J} < \infty, \quad \sqrt{\lambda(J)/\eta_J} < 1 \ (J \in \mathcal{P}_1) \quad \text{and} \tag{3.19}$$

$$\{J \in \mathcal{P}_1 : \ \eta_J > \varepsilon\} \quad \text{is finite for each} \quad \varepsilon > 0. \tag{3.20}$$

Denote $d_J := \sqrt{\lambda(J)/\eta_J}$. For each point x which is an endpoint of a member of $\mathcal{P}_1$, choose $c(x) \geq 0$ such that

$$c(x) = 0 \quad \text{whenever} \quad x \notin G_1 \quad \text{and} \tag{3.21}$$

$$0 < c(x) \leq 10^{-10} \cdot (d_J)^2 \eta_J \ \text{ if } \ x \in G_1 \ \text{ is an endpoint of some } \ J \in \mathcal{P}_1. \tag{3.22}$$

Since $\sum_{J \in \mathcal{P}_1} d_J < \infty$, by Lemma 3.14 we can choose $0 < d_2 < \infty$ and an increasing homeomorphism $\Psi : [0, d_2] \to [0, d_1]$ such that Ψ^{-1} is absolutely continuous and $\lambda(\Psi^{-1}(J)) = d_J$ for each $J \in \mathcal{P}_1$. Set $\mathcal{P}_2 := \{\Psi^{-1}(J) : \ J \in \mathcal{P}_1\}$, $G_2 := \Psi^{-1}(G_1)$ and $H_2 := [0, d_2] \setminus G_2$.

Now consider an interval $I = [u,v] \in \mathcal{P}_2$. Let $J := \Psi(I)$, $\eta := \eta_J$, $d := d_J$, $V := \lambda(J)$, $c_l(I) = c_l := c(\Psi(u))$, and $c_r(I) = c_r := c(\Psi(v))$.

Since $\max(c_l, c_r) \le 10^{-10} \cdot d^2\eta$ by (3.22) and $d < 1$ by (3.19), we can choose by Lemma 3.16 an increasing C^2 homeomorphism $\varphi_I : I \to J$ such that

$$\varphi_I'(u) = c_l,\ \varphi_I'(v) = c_r,\ 0 < \varphi_I'(x) \le 19 \cdot \sqrt{\eta V} \quad (x \in \mathrm{int}(I)), \tag{3.23}$$

$$\varphi_I''(u) = \varphi_I''(v) = 0 \ \text{ and } \ |\varphi_I''(x)| \le 19 \cdot \eta \quad (x \in I). \tag{3.24}$$

Thus, setting $\varphi(x) := \varphi_I(x)$ if $x \in I \in \mathcal{P}_2$ and $\varphi(x) := \Psi(x)$ if $x \in [0, d_2] \setminus \bigcup \mathcal{P}_2$, we easily see that $\varphi : [0, d_2] \to [0, d_1]$ is an increasing homeomorphism. Further, the definition of $c_l(I)$, $c_r(I)$, (3.23), (3.24), and (3.22) easily imply that φ is C^2 on G_2 and $\varphi' > 0$ on G_2. Since $(f_1 \circ \varphi)''(x) = f_1''(\varphi(x)) \cdot (\varphi'(x))^2 + f_1'(\varphi(x)) \cdot \varphi''(x)$ for $x \in G_2$, we easily obtain that $f_1 \circ \varphi$ is locally $D^{2,\infty}$ on G_2, and $f_1 \circ \varphi$ is even C^2 on G_2, if (3.15) holds. Using (3.17), we obtain $(f_1 \circ \varphi)'(x) \ne 0$ for $x \in G_2$.

Now consider an interval $I = \Psi^{-1}(J) \in \mathcal{P}_2$ and $x \in I \cap G_2$. Then

$$(f_1 \circ \varphi)''(x) = (f_1 \circ \varphi_I)''(x) = f_1''(\varphi_I(x)) \cdot (\varphi_I'(x))^2 + f_1'(\varphi_I(x)) \cdot \varphi_I''(x). \tag{3.25}$$

Consequently, (recall that $\eta = \eta_J$ and $V = \lambda(J)$) by (3.17), (3.18), (3.23), (3.24), and (3.19) we have

$$\|(f_1 \circ \varphi)''(x)\| \le \frac{K}{V} \cdot 19^2 \cdot \eta_J V + 19 \cdot \eta_J \le \eta_J (19^2 \cdot K + 19) \quad \text{and} \tag{3.26}$$

$$\|(f_1 \circ \varphi)'(x)\| = \|f_1'(\varphi_I(x)) \cdot \varphi_I'(x)\| \le 19 \cdot \sqrt{\eta_J \lambda(J)}. \tag{3.27}$$

Now we will show that

$$\text{for each } \ z \in H_2,\ (f_1 \circ \varphi)''(z) = 0 \text{ and } (f_1 \circ \varphi)'' \text{ is continuous at } z. \tag{3.28}$$

First, we will prove that $(f_1 \circ \varphi)_+'(z) = 0$ for each $z \in H_2 \setminus \{d_2\}$. If z is the left endpoint of some $I \in \mathcal{P}_2$, we obtain $(f_1 \circ \varphi)_+'(z) = 0$ easily from the first equality of (3.27), (3.21) and the first equality of (3.23). If z is not the left endpoint of any $I \in \mathcal{P}_2$, consider an arbitrary $\varepsilon > 0$. Using (3.20) and (3.27), we easily see that there exists $\delta > 0$ such that $\|(f_1 \circ \varphi)'(x)\| < \varepsilon$ for each $x \in (z, z+\delta) \cap G_2$. Since clearly $\mathcal{H}^1((f_1 \circ \varphi)(H_2)) = \mathcal{H}^1(f(H)) = 0$, using the mean value theorem and Lemma 2.6(iii), we easily obtain that $f_1 \circ \varphi$ is ε-Lipschitz on $[z, z+\delta]$. Thus $(f_1 \circ \varphi)_+'(z) = 0$. Similarly we obtain that $(f_1 \circ \varphi)_-'(z) = 0$ for each $z \in H_2 \setminus \{0\}$.

To prove (3.28), consider a point $z \in H_2 \setminus \{d_2\}$ and observe that,

$$\begin{aligned}&\text{for each } \varepsilon > 0, \text{ there exists a } \delta > 0 \text{ such that}\\ &\|(f_1 \circ \varphi)''(x)\| < \varepsilon \quad \text{for each } \ x \in (z, z+\delta) \cap G_2.\end{aligned} \tag{3.29}$$

Indeed, if z is not the left endpoint of some $I \in \mathcal{P}_2$, then (3.29) follows from (3.20) and (3.26). If z is the left endpoint of some $I \in \mathcal{P}_2$, we obtain (3.29) easily from (3.25), (3.17), (3.18), (3.21), and the equalities of (3.23) and (3.24). Now consider an arbitrary $y \in (z, z+\delta) \cap G_2$; let w be the left endpoint of the component of G_2 which contains y. Then $(f_1 \circ \varphi)'(w) = 0$ and thus the mean value theorem and (3.29) imply $\|(f_1 \circ \varphi)'(y)\| \le \varepsilon|y - w| \le \varepsilon|y - z|$. Since $(f_1 \circ \varphi)'(y) = 0$ for each $y \in (z, z+\delta) \setminus G_2$, we see that $\lim_{y \to z+} (f_1 \circ \varphi)'(y)/|y - z| = 0$. Also using a symmetrical argument, we conclude that $(f_1 \circ \varphi)''(z) = 0$ for each $z \in H_2$. The rest of (3.28) now follows from (3.29) and its symmetrical version.

We have proved that $(f_1 \circ \varphi)''$ is locally bounded on G_2. Using also (3.28), we conclude that $(f_1 \circ \varphi)''$ is locally bounded on $[0, d_2]$, which implies that $f_1 \circ \varphi$ is $D^{2,\infty}$ on $[0, d_2]$. If (3.15) holds, we have proved that $f_1 \circ \varphi$ is C^2 on G_2; so (3.28) yields that $f_1 \circ \varphi$ is C^2 on $[0, d_2]$.

Thus, to finish the proof of the first part of the assertion, it is sufficient to define $h := \xi \circ \varphi \circ \pi$, where $\pi(x) = d_2 x,\ x \in [0,1]$.

To prove the second part of the assertion, suppose that f is nonconstant on any interval. Then $v^* = v_f = \xi^{-1}$; so Lemma 2.6(iv) implies $\lambda(\xi^{-1}(H)) = 0$. Since Ψ^{-1} is absolutely continuous and $\varphi^{-1}(\xi^{-1}(H)) = \Psi^{-1}(\xi^{-1}(H))$, we have $\lambda((\xi \circ \varphi)^{-1}(H)) = 0$, and thus also $\lambda(h^{-1}(H)) = 0$. □

4. C^2-PARAMETRIZATIONS

The following proposition shows that, if X admits an equivalent Fréchet smooth norm, the characterization of those $f : [0,1] \to X$, which allow a C^2 parametrization with nonzero derivatives of the first order, is quite simple.

Proposition 4.1. *Let X be a Banach space which admits an equivalent Fréchet smooth norm, let $f : [0,1] \to X$ be continuous. Then the following are equivalent.*

(i) *There exists an increasing homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is C^2 with $(f \circ h)'(x) \neq 0$ for all $x \in [0,1]$.*
(ii) *f is BV, is not constant on any interval, and $F := f \circ v_f^{-1}$ is C^2.*

Proof. Let h be as in (i) and $g := f \circ h$. It is easy to see that f is BV, is not constant on any interval, and $F := f \circ v_f^{-1} = g \circ v_g^{-1}$. So, (ii) follows from Lemma 3.2. Now suppose that (ii) holds. Set $\ell := v_f(1)$ and $h(t) := v_f^{-1}(\ell \cdot t)$, $t \in [0,1]$. Using Lemma 2.5, we see that h has the property from (i). □

Definition 4.2. For a continuous $f : [0,1] \to X$, denote by D_f the set of all points $x \in [0,1]$ for which there is no interval $(c,d) \subset [0,1]$ containing x such that $f|_{[c,d]}$ has a C^2 arc-length parametrization.

The set D_f is obviously closed and $\{0,1\} \subset D_f$. Further, if h is an increasing homeomorphism of $[0,1]$ onto itself, then clearly

$$D_{f \circ h} = h^{-1}(D_f). \tag{4.1}$$

Lemma 4.3. *Let X be a Banach space which admits an equivalent Fréchet smooth norm, let $g : [0,1] \to X$ be C^2, and let $x \in D_g$. Then either $g'(x) = 0$ or $x \in \{0,1\}$.*

Proof. Suppose that $x \in (0,1) \cap D_g$. For a contradiction, assume $g'(x) \neq 0$. Then there exists $\delta > 0$ such that $g'(y) \neq 0$ for each $y \in [x-\delta, x+\delta]$. Then Lemma 3.2 gives that $g|_{[x-\delta,x+\delta]}$ has a C^2 arc-length parametrization, which contradicts $x \in D_g$. □

The following proposition states the main necessary conditions for the existence of a C^2 parametrization.

Proposition 4.4. *Let X be a Banach space which admits an equivalent Fréchet smooth norm and suppose that a nonconstant continuous $f : [0,1] \to X$ admits a C^2 parametrization. Set $G := (0,1) \setminus D_f$. Then f is BV and the following conditions hold.*

(i) *$W^\delta(f,G) < \infty$ for each $\delta > 0$.*
(ii) *$\sum_{I \in \mathcal{I}} \sqrt{V(f,I)} < \infty$, where $\mathcal{I}$ is the family of all components of G.*
(iii) *$\int_{v_f(G)} \sqrt{\|F''\|} < \infty$, where $F := \mathcal{A}_f$ (see Definition 2.1).*
(iv) *$\mathcal{H}^1(f(D_f)) = 0$.*

Proof. Obviously, there exists an increasing homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is a C^2 function. Since $f \circ h$ is BV, clearly f is BV as well. Put $g := f \circ h$ and $G^* = h^{-1}(G)$. By (4.1), $D_g = h^{-1}(D_f)$ and $G^* = [0,1] \setminus D_g$. Lemma 4.3 implies that $g'(x) = 0$ for each $x \in D_g \cap (0,1) = (0,1) \setminus G^*$. Thus g fulfils the assumptions of Lemma 3.6 (with $f := g$ and $G := G^*$). Therefore, for each $\delta > 0$ we have $W^\delta(g, G^*) < \infty$, and so also $W^\delta(f, G) < \infty$ by Remark 3.4. So (i) is proved; (ii) and (iii) follow immediately by Remark 3.5 and Lemma 3.12(ii).

Finally, Lemma 2.4 implies $\mathcal{H}^1(g(D_g)) = \mathcal{H}^1(g(D_g \cap (0,1))) = 0$. Since $g(D_g) = f(D_f)$, we have proved (iv).

□

The main result of the present section is the following theorem which solves the C^2 parametrization problem for curves in Banach spaces admitting a Fréchet smooth norm. (Observe that condition (i) is clearly equivalent to the existence of a C^2 parametrization of f, and implies that f is BV).

Theorem 4.5. *Let X be a Banach space which admits an equivalent Fréchet smooth norm. Let a BV continuous nonconstant $f : [0,1] \to X$ be given, and put $G := [0,1] \setminus D_f$. Then the following are equivalent.*

(i) *There exists an increasing homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is a C^2 function.*
(ii) *There exists an increasing homeomorphism φ of $[0,1]$ onto itself such that $f \circ \varphi$ is C^2 and $(f \circ \varphi)'(x) \neq 0$ for each $x \in \varphi^{-1}(G)$.*
(iii) *$\mathcal{H}^1(f(D_f)) = 0$ and $W^\delta(f, G) < \infty$ for each $\delta > 0$.*
(iv) *$\mathcal{H}^1(f(D_f)) = 0$ and $W^\delta(f, G) < \infty$ for some $\delta > 0$.*
(v) *$\mathcal{H}^1(f(D_f)) = 0$ and $\sum_{I \in \mathcal{P}} \sqrt{V(f, I)} < \infty$ whenever $\mathcal{P}$ is an (f, δ, ∞)-partition of G with $\delta > 0$.*
(vi) *$\mathcal{H}^1(f(D_f)) = 0$ and $\sum_{I \in \mathcal{P}^*} \sqrt{V(f, I)} < \infty$ for some $(f, 0, K^*)$-partition $\mathcal{P}^*$ of G with $0 < K^* < \infty$.*
(vii) *There exists a family $\mathcal{S}$ of pairwise non-overlapping compact intervals such that $\operatorname{int}(J) \subset G$ for each $J \in \mathcal{S}$ and*

$$\sum_{J \in \mathcal{S}} V(f, J) = V(f, [0,1]), \quad \sum_{J \in \mathcal{S}} \sqrt{V(f, J)} < \infty, \quad \sum_{J \in \mathcal{S}} V(f, J) \cdot \sqrt{S_J} < \infty, \tag{4.2}$$

where $S_J = \sup_{y \in v_f(\operatorname{int}(J))} \|F''(y)\|$ and $F := \mathcal{A}_f$ (see Definition 2.1).

If f is nonconstant on any interval, then in (ii) we can also assert $\lambda(\varphi^{-1}(G)) = 1$.

Proof. It is sufficient to prove the implications (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (vi) $\Longrightarrow$ (vii) $\Longrightarrow$ (iii), (vi) $\Longrightarrow$ (ii), and (iii) $\Longrightarrow$ (v) $\Longrightarrow$ (vi).

Note that (since f is nonconstant) both $\mathcal{H}^1(f(D_f)) = 0$ and the equality of (4.2) imply $G \neq \emptyset$ by Lemma 2.6(i),(ii).

The implication (ii) $\Longrightarrow$ (i) is trivial and (i) $\Longrightarrow$ (iii) was proved in Proposition 4.4.

Obviously, (iii) $\Longrightarrow$ (iv). To prove (iv) $\Longrightarrow$ (vi), let $\delta > 0$ be as in condition (iv), and put $K^* := \delta$. By Lemma 3.10 we can choose an (f, δ, K^*)-partition $\mathcal{P}^*$ of G. Since $\mathcal{P}^*$ is clearly an $[f, \delta, \infty]$-partition of G, we obtain $\sum_{I \in \mathcal{P}^*} \sqrt{V(f, I)} < \infty$ by Lemma 3.12(i). Thus we obtain (vi), since $\mathcal{P}^*$ is clearly an $(f, 0, K^*)$-partition of G.

To prove (vi) $\Longrightarrow$ (vii) suppose that (vi) holds and $\mathcal{P}^*$ is given. Put $\mathcal{S} := \mathcal{P}^*$. Then the equality of (4.2) holds by Lemma 2.6(i) (applied to $G := \bigcup_{P \in \mathcal{S}} \operatorname{int}(P)$).

Since $V(f,J)\cdot\sqrt{S_J} \le \sqrt{K^*}\cdot\sqrt{V(f,J)}$ for each $J \in \mathcal{S}$, also both inequalities of (4.2) hold.

To prove (vii) $\Longrightarrow$ (iii), let $\mathcal{S}$ be as in (vii). Using Lemma 2.6(ii) (with $[a,b] := [0,1]$ and $G := \bigcup_{P\in\mathcal{S}} \operatorname{int} P$), we obtain $\mathcal{H}^1(f(D_f)) \le \mathcal{H}^1(f([0,1]\setminus\bigcup_{P\in\mathcal{S}}\operatorname{int}(P))) = 0$. The second condition of (iii) follows immediately from Lemma 3.13.

If (vi) holds, then Lemma 3.17 (with $K := K^*$ and $H := D_f$) implies (ii) (and also $\lambda(\varphi^{-1}(G)) = 1$ if f is nonconstant on any interval).

The implication (iii) $\Longrightarrow$ (v) follows immediately from Lemma 3.12(i) and Remark 3.9. To prove (v) $\Longrightarrow$ (vi), choose by Lemma 3.10 an $(f,1,1)$-partition $\mathcal{P}$ of G. By (v), $\sum_{I\in\mathcal{P}}\sqrt{V(f,I)} < \infty$; so (vi) holds with $\mathcal{P}^* := \mathcal{P}$ and $K^* := 1$.

□

Remark 4.6.
- (i) The assumption that X admits a Fréchet smooth norm was used only in the proof of (i) $\Longrightarrow$ (iii). So, any of conditions (iii)-(vii) implies (i) and (ii) in an arbitrary X.
- (ii) The conditions (v) and (vi) give an "algorithmic" way how to decide whether (i) holds:

  Decide whether $\mathcal{H}^1(f(D_f)) = 0$. If it holds, then choose an (f,δ,K)-partition $\mathcal{P}$ of $G := [0,1]\setminus D_f$ with $\delta > 0$ and $K < \infty$ (such a partition exists by Lemma 3.10) and decide whether $\sum_{I\in\mathcal{P}}\sqrt{V(f,I)} < \infty$.
- (iii) The condition (vii) needs no auxiliary notions for its formulation. On the other hand, in concrete situations, it is not suitable for a proof that (i) does not hold.

Remark 4.7. Let X be a Banach space which admits an equivalent Fréchet smooth norm. Let $f : [0,1] \to X$ be BV continuous and $G := [0,1]\setminus D_f$. Then the following are equivalent.

- (a) *There exists an increasing homeomorphism φ of $[0,1]$ onto itself such that $f\circ\varphi$ is C^2 and $(f\circ\varphi)'(x) \ne 0$ almost everywhere.*
- (b) *f is nonconstant on any interval and any of conditions $(iii)-(vii)$ holds.*

It follows immediately from Theorem 4.5 and the simple observation that (a) implies that f is nonconstant on any interval.

Proposition 4.8. *Let X be a Banach space which admits an equivalent Fréchet smooth norm. Assume that $f : [0,1] \to X$ is continuous, BV and nonconstant on any interval. Let $F := f\circ v_f^{-1}$ and $\ell := v_f(1)$. Suppose that F'' is continuous on $(0,\ell)$ and there exists $\delta > 0$ such that $\|F''\|$ is monotone on $(0,\delta)$ and on $(\ell-\delta,\ell)$.*

Then the following are equivalent.

- (i) $\int_0^\ell \sqrt{\|F''(t)\|}\,dt < \infty$,
- (ii) *There exists an increasing homeomorphism h of $[0,1]$ onto itself such that $f\circ h$ is C^2.*

Proof. The implication (ii) $\Longrightarrow$ (i) follows from Proposition 4.4.

Now suppose that (i) holds. Choose points $(x_i)_{i\in\mathbb{Z}}$ in $(0,\ell)$ such that $\mathcal{P} := \{[x_i,x_{i+1}] :\ i\in\mathbb{Z}\}$ is a generalized partition of $(0,\ell)$ and $x_{i+1}-x_i \sim |i|^{-4}$, $i\to\pm\infty$. Set $t_i := v_f^{-1}(x_i)$ and $\mathcal{S} := \{v_f^{-1}(I) :\ I\in\mathcal{P}\} = \{[t_i,t_{i+1}] :\ i\in\mathbb{Z}\}$. To prove (ii), it is sufficient to show that (4.2) from Theorem 4.5 holds. The first two parts of (4.2) clearly hold; thus it is sufficient to verify that

$$\sum_{J\in\mathcal{S}} V(f,J)\cdot\sqrt{S_J} < \infty, \quad \text{where} \quad S_J = \sup_{y\in v_f(\operatorname{int}(J))}\|F''(y)\|. \tag{4.3}$$

Choose $p \in \mathbb{N}$ such that $x_p > \ell - \delta$ and $x_{-p} < \delta$. First suppose that $\|F''\|$ is nondecreasing on $(\ell - \delta, \ell)$. Then, for each $i \geq p$,

$$\int_{x_{i+1}}^{x_{i+2}} \sqrt{\|F''(t)\|}\, dt \geq (x_{i+2} - x_{i+1}) \sqrt{\|F''(x_{i+1})\|} \geq V(f, [t_{i+1}, t_{i+2}]) \sqrt{S_{[t_i, t_{i+1}]}}.$$

Consequently $\sum_{i=p}^{\infty} V(f, [t_{i+1}, t_{i+2}]) \sqrt{S_{[t_i, t_{i+1}]}} < \infty$, and since $V(f, [t_{i+1}, t_{i+2}]) \sim (i+1)^{-4} \sim i^{-4} \sim V(f, [t_i, t_{i+1}])$, $i \to \infty$, we obtain

$$\sum_{i=p}^{\infty} V(f, [t_i, t_{i+1}]) \sqrt{S_{[t_i, t_{i+1}]}} < \infty. \tag{4.4}$$

If $\|F''\|$ is nonincreasing on $(\ell - \delta, \ell)$, then $\|F''\|$ is bounded on $(\ell - \delta, \ell)$, and thus (4.4) clearly also holds. By a quite symmetrical way we obtain

$$\sum_{i=p}^{\infty} V(f, [t_{-(i+1)}, t_{-i}]) \sqrt{S_{[t_{-(i+1)}, t_{-i}]}} < \infty,$$

and therefore (4.3) holds. □

Example 6.5 shows that the assumption on the monotonicity of $\|F''\|$ cannot be omitted in Proposition 4.8.

Now we will state a simple consequence of our results, which might be interesting from the point of view of differential geometry.

Proposition 4.9. *Let $\ell \in (0, \infty)$ and $f : (0, \ell) \to \mathbb{R}^n$ be a C^2 curve parametrized by the arc-length. Let $\kappa_1(s) = \|f''(s)\|$, $s \in (0, \ell)$, be the first curvature of f. Let $0 < \delta < \infty$ be given.*

Then the following are equivalent:

- (i) *f admits a C^2 parametrization $g : (0,1) \to \mathbb{R}^n$ such that $g'(x) \neq 0$, $x \in (0,1)$, and g'' is bounded.*
- (ii) *$V_f^\delta := \sup\{\sum_{k=1}^n \sqrt{\lambda(I_k)}\} < \infty$, the supremum being taken over all systems $(I_k)_{k=1}^n$ of pairwise nonoverlapping compact subintervals of $(0, \ell)$ such that $\lambda(I_k) \max_{s \in I_k} \kappa_1(s) \geq \delta$, $k = 1, \ldots, n$.*

If (i) holds, then

- (iii) *$\int_0^\ell \sqrt{\kappa_1(s)}\, ds < \infty$.*

In the case when κ_1 is monotone on some right neigbourhood of 0 and on some left neighbourhood of ℓ, all conditions (i), (ii) and (iii) are pairwise equivalent.

Proof. Since f is parametrized by the arc-length, it is 1-Lipschitz; therefore there exists a 1-Lipschitz extension $\hat{f} : [0, \ell] \to X$ of f.

Now consider an arbitrary parametrization $p : [0,1] \to X$ of $\hat{f}$. Then clearly $\mathcal{A}_p = \hat{f}$. So, using only definitions of V_f^δ and $W^\delta(p, (0,1))$, we easily obtain

$$V_f^\delta \leq W^\delta(p, (0,1)) \leq V_f^\delta + 2\sqrt{V(p, [0,1]} = V_f^\delta + 2\sqrt{\ell}. \tag{4.5}$$

Now suppose that (i) holds, and let $h : (0,1) \to (0, \ell)$ be an increasing homeomorphism such that $g = f \circ h$. Let $\hat{h} : [0,1] \to [0, \ell]$ be the homeomorphism which extends h, and set $p := \hat{f} \circ \hat{h}$. Then p is clearly continuous and BV. Since $\mathcal{A}_p = \hat{f}$, we can clearly use Lemma 3.6 (with $f := p$ and $G := (0,1)$) and obtain $W^\delta(p, (0,1)) < \infty$. Now (4.5) implies (ii). Moreover, using Lemma 3.12(ii) (with $f := p$ and $G := (0,1)$), we obtain (iii).

If (ii) holds, choose a parametrization $p : [0,1] \to X$ of $\hat{f}$. Since $D_p = \{0,1\}$, and $W^\delta(p,(0,1)) < \infty$ by (4.5), we obtain that p admits a C^2 parametrization $\hat{g}$ such that $(\hat{g})'(x) \neq 0$ for each $x \in (0,1)$ by Theorem 4.5 (the equivalence of (ii) and (iv)). Setting $g := \hat{g}|_{(0,1)}$, we obtain (i).

Finally, assume that (iii) holds and that κ_1 is monotone on $(0,\sigma)$ and $(\ell-\sigma,\ell)$ for some $\sigma > 0$. Let $p : [0,1] \to X$ be a parametrization of $\hat{f}$. Using Proposition 4.8 (with $f := p$) and Theorem 4.5 (the equivalence of (i) and (ii)), we easily obtain (i). □

Remark 4.10. The proof shows that (i) is equivalent to the condition

(iv) There exists a C^2 smooth $q : [0,1] \to \mathbb{R}^n$ such that $q'(x) \neq 0$, $x \in (0,1)$, and $q|_{(0,1)}$ is a parametrization of f.

5. Parametrizations with a bounded second derivative

Both the results and proofs in the $D^{2,\infty}$ problem are quite analogous to those in the C^2 problem. The only two differences are that we now work with $[f,\delta,K]$-partitions instead of (f,δ,K)-partitions, and with the "singular set" $\widetilde{D}_f$ instead of D_f.

Making these changes (and several other obvious small changes) in the proofs of Proposition 4.1, Lemma 4.3, Proposition 4.4, Theorem 4.5 and Proposition 4.8, we obtain the proofs of analogous Proposition 5.1, Lemma 5.3, Proposition 5.4, Theorem 5.5 and Proposition 5.6 below. So, we omit the proofs of these results.

The last result of the present section (Proposition 5.7) describes one situation when the $D^{2,\infty}$ problem is equivalent to the C^2 problem.

Proposition 5.1. *Let X be a Banach space which admits an equivalent Fréchet smooth norm, let $f : [0,1] \to X$ be continuous. Then the following are equivalent.*

(i) *There exists an increasing homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is $D^{2,\infty}$ with $(f \circ h)'(x) \neq 0$ for all $x \in [0,1]$.*

(ii) *f is BV, is not constant on any interval, and $F := f \circ v_f^{-1}$ is $D^{2,\infty}$.*

Definition 5.2. For a continuous $f : [0,1] \to X$, denote by $\widetilde{D}_f$ the set of all points $x \in [0,1]$ for which there is no interval $(c,d) \subset [0,1]$ containing x such that $f|_{[c,d]}$ has a $D^{2,\infty}$ arc-length parametrization.

The set $\widetilde{D}_f$ is obviously closed and $\{0,1\} \subset \widetilde{D}_f$. Further, if h is an increasing homeomorphism of $[0,1]$ onto itself, then clearly

$$\widetilde{D}_{f\circ h} = h^{-1}(\widetilde{D}_f). \tag{5.1}$$

Lemma 5.3. *Let X be a Banach space which admits an equivalent Fréchet smooth norm, let $g : [0,1] \to X$ be $D^{2,\infty}$, and let $x \in \widetilde{D}_g$. Then either $g'(x) = 0$ or $x \in \{0,1\}$.*

Proposition 5.4. *Let X be a Banach space which admits an equivalent Fréchet smooth norm and suppose that a nonconstant continuous $f : [0,1] \to X$ admits an equivalent $D^{2,\infty}$ parametrization. Set $G := (0,1) \setminus \widetilde{D}_f$. Then f is BV and the following conditions hold.*

(i) *$W^\delta(f,G) < \infty$ for each $\delta > 0$.*

(ii) *$\sum_{I\in\mathcal{I}} \sqrt{V(f,I)} < \infty$, where $\mathcal{I}$ is the family of all components of G.*

(iii) $\int_{v_f(G)} \sqrt{\|F''\|} < \infty$, *where* $F := \mathcal{A}_f$ *(see Definition 2.1).*
(iv) $\mathcal{H}^1(f(\widetilde{D}_f)) = 0$.

Theorem 5.5. *Let X be a Banach space which admits an equivalent Fréchet smooth norm. Let a BV continuous nonconstant $f : [0,1] \to X$ be given, and put $G := [0,1] \setminus \widetilde{D}_f$. Then the following are equivalent.*

(i) *There exists an increasing homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is a $D^{2,\infty}$ function.*
(ii) *There exists an increasing homeomorphism φ of $[0,1]$ onto itself such that $f \circ \varphi$ is $D^{2,\infty}$ and $(f \circ \varphi)'(x) \neq 0$ for each $x \in \varphi^{-1}(G)$.*
(iii) *$\mathcal{H}^1(f(\widetilde{D}_f)) = 0$ and $W^\delta(f, G) < \infty$ for each $\delta > 0$.*
(iv) *$\mathcal{H}^1(f(\widetilde{D}_f)) = 0$ and $W^\delta(f, G) < \infty$ for some $\delta > 0$.*
(v) *$\mathcal{H}^1(f(\widetilde{D}_f)) = 0$ and $\sum_{I \in \mathcal{P}} \sqrt{V(f, I)} < \infty$ whenever $\mathcal{P}$ is an $[f, \delta, \infty]$-partition of G with $\delta > 0$.*
(vi) *$\mathcal{H}^1(f(\widetilde{D}_f)) = 0$ and $\sum_{I \in \mathcal{P}^*} \sqrt{V(f, I)} < \infty$ for some $[f, 0, K^*]$-partition $\mathcal{P}^*$ of G with $K^* < \infty$.*
(vii) *There exists a family $\mathcal{S}$ of pairwise non-overlapping compact intervals such that $\operatorname{int}(J) \subset G$ for each $J \in \mathcal{S}$ and*

$$\sum_{J \in \mathcal{S}} V(f, J) = V(f, [0,1]), \quad \sum_{J \in \mathcal{S}} \sqrt{V(f, J)} < \infty, \quad \sum_{J \in \mathcal{S}} V(f, J) \cdot \sqrt{S_J} < \infty, \tag{5.2}$$

where $S_J = \sup_{y \in v_f(\operatorname{int}(J))} \|F''(y)\|$ and $F := \mathcal{A}_f$ (see Definition 2.1).

If f is nonconstant on any interval, then in (ii) we can also assert $\lambda(\varphi^{-1}(G)) = 1$.

Let us note that the remarks quite analogous to Remark 4.6 and Remark 4.7 (we only replace C^2 by $D^{2,\infty}$, (f, δ, K) by $[f, \delta, K]$, and D_f by $\widetilde{D}_f$) are true.

Proposition 5.6. *Let X be a Banach space which admits an equivalent Fréchet smooth norm. Assume that $f : [0,1] \to X$ is continuous, BV and nonconstant on any interval. Let $F := f \circ v_f^{-1}$ and $\ell := v_f(1)$. Suppose that F'' is locally bounded on $(0, \ell)$ and for some $\delta > 0$ we have that $\|F''\|$ is monotone on $(0, \delta)$ and on $(\ell - \delta, \ell)$. Then the following are equivalent.*

(i) $\int_0^\ell \sqrt{\|F''(t)\|}\, dt < \infty$,
(ii) *There exists an increasing homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is $D^{2,\infty}$.*

Our main results easily imply the following proposition, which should be compared with Example 6.1 and Example 6.2 below.

Proposition 5.7. *Let $f : [0,1] \to X$ be continuous and BV such that $B := D_f \setminus \widetilde{D}_f$ is finite. Then f admits a C^2 parametrization if and only if f admits a $D^{2,\infty}$ parametrization.*

Proof. The "only if" implication is obvious. So, suppose that f admits a $D^{2,\infty}$ parametrization and choose a family $\mathcal{S}$ by condition (vii) of Theorem 5.5. Dividing members of $\mathcal{S}$ by points of B, we obtain a non-overlapping system $\mathcal{S}^*$ of compact intervals (such that $\bigcup_{I \in \mathcal{S}^*} \operatorname{int}(I) \subset \bigcup_{J \in \mathcal{S}} \operatorname{int}(J)$ and $\bigcup_{I \in \mathcal{S}} \operatorname{int}(I) \setminus \bigcup_{J \in \mathcal{S}^*} \operatorname{int}(J) = B \cap \bigcup_{I \in \mathcal{S}} \operatorname{int}(I)$). Since $\mathcal{S}^*$ clearly shows that the condition (vii) of Theorem 4.5 holds, f admits a C^2 parametrization by Theorem 4.5. □

6. Examples

Example 6.1. There exists a function $f : [0,1] \to \mathbb{R}^2$ which admits a $D^{2,\infty}$ parametrization, does not admits a C^2 parametrization and $D_f \setminus \widetilde{D}_f = \{1/n : \ n = 2,3,\dots\}$.

Proof. Set $C := \{1/n : \ n = 2,3,\dots\} \cup \{0,1\}$. By Lemma 2.7 there exists a real function φ on $[0,1]$ which has a bounded derivative on $[0,1]$, and C is the set of discontinuity of φ'. Now identify $\mathbb{R}^2$ with the complex plane and set $f(x) := \int_0^x e^{i\varphi(t)}\, dt$. Then f is Lipschitz, $\|f'(x)\| = 1$, $x \in [0,1]$, and thus f is parametrized by the arc-length by (2.2). Since $f''(t) = ie^{i\varphi(t)}\varphi'(t)$ for $t \in (0,1)$, we easily see that $\widetilde{D}_f = \{0,1\}$ and $D_f = C$. Since the condition (vii) of Theorem 5.5 clearly holds with $\mathcal{S} := \{[0,1]\}$, f admits a $D^{2,\infty}$ parametrization. On the other hand, $\sum_{n\in\mathbb{N}} \sqrt{V(f,[1/(n+1),1/n])} = \sum_{n\in\mathbb{N}} \sqrt{1/(n(n+1))} = \infty$. So, the condition (ii) of Proposition 4.4 does not hold, and consequently f does not admit a C^2 parametrization. □

Example 6.2. There exists a function $f : [0,1] \to \mathbb{R}^2$ which admits a C^2 parametrization and $D_f \setminus \widetilde{D}_f$ is uncountable.

Proof. Let $C \subset [0,1]$ be a closed set which is constructed by the same way as the Cantor ternary set, with the only difference that in the n-th step of the construction from any of 2^{n-1} closed intervals I we delete a concentric open interval of the length $(3/5)\lambda(I)$. So, $H := (0,1) \setminus C$ has, for each $n \in \mathbb{N}$, 2^{n-1} components of length $(3/5)(1/5)^{n-1}$ (and no other component). So, $\lambda(C) = 1 - \sum_{n=1}^{\infty}(3/5)(2/5)^{n-1} = 0$. By Lemma 2.7 there exists a real function φ on $[0,1]$ which has a bounded derivative on $[0,1]$, and C is the set of discontinuity of φ'. Setting $f(x) := \int_0^x e^{i\varphi(t)}\, dt$, we obtain as in Example 6.1 that f is parametrized by the arc-length, $\widetilde{D}_f = \{0,1\}$ and $D_f = C$. To prove that f admits a C^2 parametrization, we will verify condition (vii) of Theorem 4.5. We will show that (4.2) holds, if we define $\mathcal{S}$ as the system of the closures of all components of H. First,

$$\sum_{J\in\mathcal{S}} V(f,J) = \sum_{J\in\mathcal{S}} \lambda(J) = 1 = V(f,[0,1]).$$

Further,

$$\sum_{J\in\mathcal{S}} \sqrt{V(f,J)} = \sum_{n=1}^{\infty} 2^{n-1}\sqrt{(3/5)(1/5)^{n-1}} < \infty.$$

Finally, the last part of (4.2) also holds, since $S_J = 1$ for each $J \in \mathcal{S}$. □

Example 6.3. For $s > 0$, consider the spiral $f : [0,1] \to \mathbb{R}^2$ defined by $f(0) = 0$ and

$$f(t) = (x(t), y(t)) = (t^s\cos(1/t), t^s\sin(1/t)), \quad 0 < t \le 1.$$

Identifying $\mathbb{R}^2$ with the complex plane, we have, for $t \in (0,1)$,

$$f(t) = t^s e^{i/t} \quad \text{and} \quad f'(t) = e^{i/t}t^{s-2}(st - i).$$

Consequently, $\|f'(t)\| \sim t^{s-2},\ t \to 0+$. So, using (2.2) on intervals $[\delta,1]$, we easily obtain that f is BV if and only if $s > 1$. Using the well-known formula for the oriented curvature $k(t)$ (see e.g. [S, p. 26]) we obtain

$$k(t) = \frac{\det(f'(t), f''(t))}{\|f'(t)\|^3} = \frac{s(1-s)t^{2s-4} - t^{2s-6}}{t^{3s-6}(1+s^2t^2)^{3/2}} < 0, \quad t \in (0,1). \tag{6.1}$$

Denote $F := f \circ v_f^{-1}$ and $\ell := v_f(1)$. Since $\|F''(v_f(t))\| = |k(t)|$ and $v_f(t)$ is increasing, (6.1) easily implies that there is a $\delta > 0$ such that $\|F''\|$ is monotone on $(0, \delta)$ and on $(\ell - \delta, \ell)$. Since $v_f'(t) = \|f'(t)\|$, we have

$$\int_0^\ell \sqrt{\|F''(y)\|}\, dy = \int_0^1 \sqrt{|k(t)|}\, \|f'(t)\|\, dt.$$

Using (6.1), we easily obtain $\sqrt{|k(t)|}\, \|f'(t)\| \sim t^{s/2-2}$, $t \to 0+$. Consequently, using Proposition 4.8 (resp. Proposition 5.6), we obtain that f admits a C^2 (resp. $D^{2,\infty}$) parametrization if and only if $s > 2$.

In the following example, we need the following well-known fact.

Lemma 6.4. *Let $k : (0,1) \to \mathbb{R}$ be positive and C^∞. Then there is a continuous $f : [0,1] \to \mathbb{R}^2$ parametrized by the arc-length, C^∞ on $(0,1)$, and such that $\|f''(x)\| = k(x)$ for $x \in (0,1)$.*

Proof. By the Fundamental Theorem of the local theory of curves (see e.g. [Ku, Theorem 2.15]), there exists $g : (0,1) \to \mathbb{R}^2$ parametrized by the arc-length, which is C^∞, and $\|g''(x)\| = k(x)$ for $x \in (0,1)$. Since g is 1-Lipschitz, it has a continuous extension f to $[0,1]$, which has all the desired properties. □

Example 6.5. There exists a continuous $f : [0,1] \to \mathbb{R}^2$ which is parametrized by the arc-length, is C^∞ on $(0,1)$, does not allow a $D^{2,\infty}$-parametrization, but $\int_0^1 \sqrt{\|f''\|} < \infty$.

Proof. Let $\mathcal{P} = \{I_n : n \in \mathbb{N}\}$ be a generalized partition of $(0,1)$ such that $\lambda(I_n) = c/n^2$ for some $c > 0$. Choose closed intervals $J_n \subset I_n$ with $\lambda(J_n) = c/n^4$. Clearly, we can choose a positive C^∞ function $k : (0,1) \to (0,\infty)$ such that, for each $n \in \mathbb{N}$, we have $\max_{x \in J_n} k(x) = n^4$ and $k(x) \le 1$ for $x \in I_n \setminus J_n$. Choose an f corresponding to k by Lemma 6.4. We easily see that $\mathcal{P}$ is an (f, c, ∞)-partition of $(0,1)$, but $\sum_{I \in \mathcal{P}} \sqrt{V(f,I)} = \sum_{I \in \mathcal{P}} \sqrt{\lambda(I)} = \infty$. So, condition (v) of Theorem 5.5 does not hold (see Remark 3.9), and therefore f does not allow a $D^{2,\infty}$-parametrization. On the other hand, $\int_0^1 \sqrt{\|f''\|} \le 1 + \sum (c/n^4)\sqrt{n^4} < \infty$. □

7. The case of real valued functions

As we noted in Introduction, the case of "higher order smooth" parametrizations for $X = \mathbb{R}$ was settled independently by Laczkovich and Preiss [LP] and Lebedev [L]. Both papers contain (formally slightly different) characterizations of those $f : [0,1] \to \mathbb{R}$ which allow an equivalent C^n ($n \in \mathbb{N}$) parametrization (or a parametrization with bounded n-th derivative). Lebedev's results give that for a continuous $f : [0,1] \to \mathbb{R}$ the following conditions are equivalent:

(i) f admits a C^n parametrization.
(ii) f admits a parametrization with bounded n-th derivative.
(iii)
$$\lambda(f(K_f)) = 0 \ \text{ and } \sum_{\alpha \in A} (\omega_\alpha^f)^{1/n} < \infty,$$
where $(I_\alpha)_{\alpha \in A}$ are all maximal open intervals in $[0,1]$ on which f is constant or strictly monotone, $K_f := [0,1] \setminus \bigcup_{\alpha \in A} I_\alpha$ is the set of points "of varying monotonicity" of f (denoted by M_f in [L]), and ω_α^f is the oscillation of f on I_α.

Laczkovich and Preiss [LP] show that the conditions (i) and (ii) are equivalent to

(iv)

$$V_{1/n}(f, K_f) < \infty,$$

where

$$V_{1/n}(f, K_f) := \sup\{\sum_{i=1}^{m} |f(d_i) - f(c_i)|^{1/n}\},$$

the supremum being taken over all systems $[c_i, d_i]$, $i = 1, \dots, m$, of pairwise non-overlapping subintervals of $[0, 1]$ with $c_i, d_i \in K_f$.

We will indicate how, *in the case* $n = 2$, the equivalence of conditions (i), (ii) and (iii) follows from the results of the present article (without using any result of [LP] and [L]). First we will show that

$$K_f \cup E_f = \widetilde{D}_f = D_f \quad \text{for each continuous} \quad f : [0, 1] \to \mathbb{R}, \tag{7.1}$$

where E_f is the maximal open subset of $(0, 1)$ on which f is locally constant. To prove $(K_f \cup E_f) \subset \widetilde{D}_f$, suppose that $x \in [0, 1] \setminus \widetilde{D}_f$. By Definition 5.2 there is an interval $(c, d) \subset [0, 1]$ containing x such that $f|_{[c,d]}$ has a $D^{2,\infty}$ arc-length parametrization f^*. Since $f^* \in C^1$, we obtain that $|f^*| = 1$ on the domain of f^* by Lemma 2.5. Thus f^* is affine with the slope 1 or -1. Consequently, f is strictly monotone on (c, d), which implies $x \notin K_f \cup E_f$. The inclusion $\widetilde{D}_f \subset D_f$ is obvious. To prove $D_f \subset K_f \cup E_f$, suppose that $x \in [0, 1] \setminus (K_f \cup E_f)$. Then there is an interval $(c, d) \subset [0, 1]$ containing x such that f is strictly monotone on $[c, d]$. Since $V(f, [c, x]) = \pm(f(x) - f(c))$ for $x \in [c, d]$, we easily obtain that each arc-length parametrization of $f|_{[c,d]}$ is affine with the slope 1 or -1. So, $x \notin D_f$.

The implication (i) $\Longrightarrow$ (ii) is trivial.

Now suppose that (ii) holds for $n = 2$. By Proposition 5.4(iv) and (7.1) we obtain $\lambda(f(K_f)) = 0$. Let $(I_\alpha)_{\alpha \in A}$ be as in (iii) and let A^* be the set of those $\alpha \in A$, for which f is strictly monotone on I_α. By (7.1) we see that $(I_\alpha)_{\alpha \in A^*}$ is the system of all components of $(0, 1) \setminus \widetilde{D}_f$. So, using Proposition 5.4(ii), we obtain $\sum_{\alpha \in A} (\omega_\alpha^f)^{1/2} = \sum_{\alpha \in A^*} (\omega_\alpha^f)^{1/2} < \infty$. Thus we have proved (iii).

Finally suppose that (iii) holds for $n = 2$, and let I_α, $\alpha \in A$, be as in (iii). Since $\omega_\alpha^f = V(f, \overline{I_\alpha})$, we obtain $\sum_{\alpha \in A} V(f, \overline{I_\alpha}) < \infty$, and therefore f is BV by Lemma 2.6(i). We will prove that condition (vii) of Theorem 4.5 is satisfied. Let $\mathcal{S}$ be the family of those $\overline{I_\alpha}$ on which f is not constant. The equality (7.1) gives that $\operatorname{int}(J) \subset G := [0, 1] \setminus D_f$ for each $J \in \mathcal{S}$. Since clearly $\lambda(f(E_f)) = 0$, we obtain $\lambda(f(D_f)) = \lambda(f(K_f)) = 0$ by (7.1). Consequently Lemma 2.6(i) implies $\sum_{J \in \mathcal{S}} V(f, J) = \sum_{\alpha \in A} V(f, I_\alpha) = V(f, [0, 1])$, which is the first equality of (4.2). The validity of the second equality of (4.2) immediately follows from (iii). Since f is strictly monotone on each $J \in \mathcal{S}$, we obtain (as in the proof of (7.1)) that $F := \mathcal{A}_f$ (see Definition 2.1) is affine on $v_f(J)$. So, $\mathcal{S}_J = 0$, and the third equality of (4.2) holds.

Remark 7.1. Finally note that, using Lemma 2.6(i),(ii), it is easy to give a direct proof that the Laczkovich-Preiss condition (iv) is equivalent to the Lebedev condition (iii) (for any $n \in \mathbb{N}$). However, since this proof is not short, we do not present it here.

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